

# THE GENERALIZED QUADRATIC COVARIATION FOR FRACTIONAL BROWNIAN MOTION WITH HURST INDEX LESS THAN $1/2$ \*

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ABSTRACT. Let  $B^H$  be a fractional Brownian motion with Hurst index  $0 < H < 1/2$ . In this paper we study the *generalized quadratic covariation*  $[f(B^H), B^H]^{(W)}$  defined by

$$[f(B^H), B^H]_t^{(W)} = \lim_{\varepsilon \downarrow 0} \frac{2H}{\varepsilon^{2H}} \int_0^t \left\{ f(B_{s+\varepsilon}^H) - f(B_s^H) \right\} (B_{s+\varepsilon}^H - B_s^H) s^{2H-1} ds,$$

where the limit is uniform in probability and  $x \mapsto f(x)$  is a deterministic function. We construct a Banach space  $\mathcal{H}$  of measurable functions such that the generalized quadratic covariation exists in  $L^2$  and the Bouleau-Yor identity takes the form

$$[f(B^H), B^H]_t^{(W)} = - \int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t)$$

provided  $f \in \mathcal{H}$ , where  $\mathcal{L}^H(x, t)$  is the weighted local time of  $B^H$ . This allows us to write the fractional Itô formula for absolutely continuous functions with derivative belonging to  $\mathcal{H}$ . These are also extended to the time-dependent case.

## 1. INTRODUCTION

Given  $H \in (0, 1)$ , a fractional Brownian motion (fBm) with Hurst index  $H$  is a mean zero Gaussian process  $B^H = \{B_t^H, 0 \leq t \leq T\}$  such that

$$E[B_t^H B_s^H] = \frac{1}{2} [t^{2H} + s^{2H} - |t - s|^{2H}]$$

for all  $t, s \geq 0$ . For  $H = 1/2$ ,  $B^H$  coincides with the standard Brownian motion  $B$ .  $B^H$  is neither a semimartingale nor a Markov process unless  $H = 1/2$ , so many of the powerful techniques from stochastic analysis are not available when dealing with  $B^H$ . As a Gaussian process, one can construct the stochastic calculus of variations with respect to  $B^H$ . Some surveys and complete literatures for fBm could be found in Biagini *et al* [2], Decreusefond

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and Üstünel [6], Gradinaru *et al* [14, 15], Hu [18], Mishura [19] and Nualart [23]. It is well-known that the usual quadratic variation  $[B^H, B^H]_t = 0$  for  $2H > 1$  and  $[B^H, B^H]_t = \infty$  for  $2H < 1$ , where

$$[B^H, B^H]_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t (B_{s+\varepsilon}^H - B_s^H)^2 ds$$

in probability. Clearly, we have also

$$[B^H, B^H]_t = \lim_{n \rightarrow \infty} \sum_{j=1}^n \left( B_{jt/n}^H - B_{(j-1)t/n}^H \right)^2,$$

where the limit is uniform in probability. This is inconvenient to some studies and applications for fBm. We need to find a substitution tool. Recently, Gradinaru *et al* [14] (see also [15] and the references therein) have introduced some substitution tools and studied some fine problems. They introduced firstly an Itô formula with respect to a symmetric-Stratonovich integral, which is closer to the spirit of Riemann sums limits, and defined a class of high order integrals having an interest by themselves. On the other hand, inspired by Gradinaru-Nourdin [12, 13] and Nourdin *et al* [21, 22], as the substitution tool of the quadratic variation, Yan *et al* [27] considered the *generalized quadratic covariation*, and proved its existence for  $\frac{1}{2} < H < 1$  (Thanks to the suggestions of some Scholars we use the present appellation).

**Definition 1.1.** Let  $0 < H < 1$  and let  $f$  be a measurable function on  $\mathbb{R}$ . The limit

$$(1.1) \quad \lim_{\varepsilon \downarrow 0} \frac{2H}{\varepsilon^{2H}} \int_0^t \{f(B_{s+\varepsilon}^H) - f(B_s^H)\} (B_{s+\varepsilon}^H - B_s^H) s^{2H-1} ds$$

is called the *generalized quadratic covariation* of  $f(B^H)$  and  $B^H$ , denoted by  $[f(B^H), B^H]_t^{(W)}$ , provided the limit exists uniformly in probability.

In particular, we have

$$[B^H, B^H]_t^{(W)} = t^{2H}$$

for all  $0 < H < 1$ . If  $H = \frac{1}{2}$ , the generalized quadratic covariation coincides with the usual quadratic covariation of Brownian motion  $B$ . For  $\frac{1}{2} < H < 1$ , Yan *et al* [28] showed the generalized quadratic covariation can also be defined as

$$(1.2) \quad [f(B^H), B^H]_t^{(W)} = 2H \lim_{\|\pi_n\| \rightarrow 0} \sum_{t_j \in \pi_n} (\Lambda_j)^{2H-1} \{f(B_{t_j}^H) - f(B_{t_{j-1}}^H)\} (B_{t_j}^H - B_{t_{j-1}}^H),$$

provided the limit exists uniformly in probability, where  $\pi_n = \{0 = t_0 < t_1 < \dots < t_n = t\}$  denotes an arbitrary partition of the interval  $[0, t]$  with  $\|\pi_n\| = \sup_j (t_j - t_{j-1}) \rightarrow 0$ , and  $\Lambda_j = \frac{t_j}{t_j - t_{j-1}}$ ,  $j = 1, 2, \dots, n$ . Moreover, by applying the time reversal  $\hat{B}_t^H = B_{T-t}^H$  on  $[0, T]$  and the integral

$$\int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t),$$

Yan *et al* [28] constructed a Banach space  $\mathbb{B}_H$  of measurable functions such that the generalized quadratic covariation  $[f(B^H), B^H]_t^{(W)}$  exists in  $L^2$  if  $f \in \mathbb{B}_H$ , where

$$\mathcal{L}^H(x, t) = 2H \int_0^t \delta(B_s^H - x) s^{2H-1} ds$$

is the weighted local time of fBm  $B^H$ . However, when  $0 < H < \frac{1}{2}$  the method used in Yan *et al* [27, 28] is inefficacy. In the present paper, we shall consider the generalized quadratic covariation with  $0 < H < \frac{1}{2}$ . Our start point is to consider the decomposition

$$(1.3) \quad \begin{aligned} & \frac{1}{\varepsilon^{2H}} \int_0^t \{f(B_{s+\varepsilon}^H) - f(B_s^H)\} (B_{s+\varepsilon}^H - B_s^H) ds^{2H} \\ &= \frac{1}{\varepsilon^{2H}} \int_0^t f(B_{s+\varepsilon}^H) (B_{s+\varepsilon}^H - B_s^H) ds^{2H} - \frac{1}{\varepsilon^{2H}} \int_0^t f(B_s^H) (B_{s+\varepsilon}^H - B_s^H) ds^{2H}. \end{aligned}$$

Clearly, if the modulus in expression (1.3) is  $\frac{1}{\varepsilon}$ , the decomposition is meaningless in general. For example, for  $f(x) = x$  we have

$$\begin{aligned} \frac{1}{\varepsilon} \int_0^t E [B_s^H (B_{s+\varepsilon}^H - B_s^H)] ds^{2H} &= \frac{1}{\varepsilon} \int_0^t \frac{1}{2} [(s+\varepsilon)^{2H} - s^{2H} - \varepsilon^{2H}] ds^{2H} \\ &\longrightarrow -\infty, \end{aligned}$$

as  $\varepsilon \downarrow 0$ . However,

$$\begin{aligned} & \frac{1}{\varepsilon^{2H}} \int_0^t |EB_s^H (B_{s+\varepsilon}^H - B_s^H)| ds^{2H} \\ &= \frac{1}{\varepsilon^{2H}} \int_0^t \frac{1}{2} [s^{2H} + \varepsilon^{2H} - (s+\varepsilon)^{2H}] ds^{2H} \longrightarrow \frac{1}{2} t^{2H}, \end{aligned}$$

as  $\varepsilon \downarrow 0$ . Thus, for  $0 < H < \frac{1}{2}$  we can consider the decomposition (1.3). By estimating the two terms of the right hand side in the decomposition (1.3), respectively, we can construct a Banach space  $\mathcal{H}$  of measurable functions  $f$  on  $\mathbb{R}$  such that  $\|f\|_{\mathcal{H}} < \infty$ , where

$$\|f\|_{\mathcal{H}} = \sqrt{\int_0^T \int_{\mathbb{R}} |f(x)|^2 e^{-\frac{x^2}{2s^{2H}}} \frac{dx ds}{\sqrt{2\pi} s^{1-H}}} + \sqrt{\int_0^T \int_{\mathbb{R}} |f(x)|^2 e^{-\frac{x^2}{2s^{2H}}} \frac{dx ds}{\sqrt{2\pi} (T-s)^{1-H}}}.$$

We show that *generalized quadratic covariation*  $[f(B^H), B^H]_t^{(W)}$  exists in  $L^2$  for all  $t \in [0, T]$  if  $f \in \mathcal{H}$ . This allows us to write Itô's formula for absolutely continuous functions with derivative belonging to  $\mathcal{H}$  and to give the Bouleau-Yor identity. It is important to note that the decomposition (1.3) is inefficacy for  $\frac{1}{2} < H < 1$ .

This paper is organized as follows. In Section 2 we present some preliminaries for fBm. In Section 3, we establish some technical estimates associated with fractional Brownian motion with  $0 < H < \frac{1}{2}$ . In Section 4, we prove the existence of the generalized quadratic covariation. We construct the Banach space  $\mathcal{H}$  such that the generalized quadratic covariation  $[f(B^H), B^H]^{(W)}$  exists in  $L^2$  for  $f \in \mathcal{H}$ . As an application we show that the Itô type formula (Föllmer-Protter-Shiryaev's formula)

$$F(B^H) = F(0) + \int_0^t f(B_s^H) dB_s^H + \frac{1}{2} [f(B^H), B^H]_t^{(W)}$$

holds, where  $F$  is an absolutely continuous function with the derivative  $F' = f \in \mathcal{H}$ . In Section 5, we introduce the integral of the form

$$(1.4) \quad \int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t),$$

where  $x \mapsto f(x)$  is a deterministic function. We show that the integral (1.4) exists in  $L^2$ , and the Bouleau-Yor identity takes the form

$$[f(B^H), B^H]_t^{(W)} = - \int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t)$$

provided  $f \in \mathcal{H}$ . Moreover, by applying the integral (1.4) we show that (1.1) and (1.2) coincide for  $0 < H < \frac{1}{2}$  when  $f \in \mathcal{H}$ . In Section 6, we consider the time-dependent case, and define the local time of  $B^H$  with  $0 < H < \frac{1}{2}$  on a continuous curve.

## 2. PRELIMINARIES

In this section, we briefly recall some basic definitions and results of fBm. For more aspects on these material we refer to Biagini *et al* [2], Hu [18], Mishura [19], Nualart [23] and the references therein. Throughout this paper we assume that  $0 < H < \frac{1}{2}$  is arbitrary but fixed and let  $B^H = \{B_t^H, 0 \leq t \leq T\}$  be a one-dimensional fBm with Hurst index  $H$  defined on  $(\Omega, \mathcal{F}, P)$ . Let  $(\mathcal{S})^*$  be the Hida space of stochastic distributions and let  $\diamond$  denote the Wick product on  $(\mathcal{S})^*$ . Then  $t \mapsto B_t^H$  is differentiable in  $(\mathcal{S})^*$ . Denote

$$W_t^{(H)} = \frac{dB_t^H}{dt} \in (\mathcal{S})^*.$$

We call  $W^{(H)}$  the fractional white noise. For  $u : \mathbb{R}_+ \rightarrow (\mathcal{S})^*$ , in a white noise setting we define its Wick-Itô-Skorohod (WIS) stochastic integral with respect to  $B^H$  by

$$(2.1) \quad \int_0^t u_s dB_s^H := \int_0^t u_s \diamond W_s^{(H)} ds,$$

whenever the last integral exists as an integral in  $(\mathcal{S})^*$ . We call these fractional Itô integrals, because these integrals share some properties of the classical Itô integral. The integral is closed in  $L^2$ , and moreover, for any  $f \in C^{2,1}(\mathbb{R} \times [0, +\infty))$  the following Itô type formula holds:

$$(2.2) \quad \begin{aligned} f(B_t^H, t) &= f(0, 0) + \int_0^t \frac{\partial}{\partial x} f(B_s^H, s) dB_s^H \\ &\quad + \int_0^t \frac{\partial}{\partial s} f(B_s^H, s) ds + H \int_0^t \frac{\partial^2}{\partial x^2} f(B_s^H, s) s^{2H-1} ds. \end{aligned}$$

The fBm  $B^H$  has a local time  $\mathcal{L}^H(x, t)$  continuous in  $(x, t) \in \mathbb{R} \times [0, \infty)$  which satisfies the occupation formula (see Geman-Horowitz [11])

$$(2.3) \quad \int_0^t \phi(B_s^H, s) ds = \int_{\mathbb{R}} dx \int_0^t \phi(x, s) \mathcal{L}^H(x, ds)$$

for every continuous and bounded function  $\phi(x, t) : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ , and such that

$$\mathcal{L}^H(x, t) = \int_0^t \delta(B_s^H - x) ds = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \lambda(s \in [0, t], |B_s^H - x| < \epsilon),$$

where  $\lambda$  denotes Lebesgue measure and  $\delta(x)$  is the Dirac delta function. Define the so-call weighted local time  $\mathcal{L}^H(x, t)$  of  $B^H$  at  $x$  as follows

$$\mathcal{L}^H(x, t) = 2H \int_0^t s^{2H-1} \mathcal{L}^H(x, ds) \equiv 2H \int_0^t \delta(B_s^H - x) s^{2H-1} ds.$$

Then the Tanaka formula

$$(2.4) \quad |B_t^H - x| = |x| + \int_0^t \text{sign}(B_s^H - x) dB_s^H + \mathcal{L}^H(x, t)$$

holds.

For  $H \in (0, 1)$  we define the operator  $M$  on  $L^2(\mathbb{R})$  as follows (see Chapter 4 in Biagini *et al* [2] and Elliott-Van der Hoek [8]):

$$Mf(x) = -\frac{\beta_H}{H - \frac{1}{2}} \frac{d}{dx} \int_{\mathbb{R}} \frac{(s - x)}{|s - x|^{\frac{3}{2}-H}} f(s) ds, \quad f \in L^2(\mathbb{R}),$$

where  $\beta_H$  is a normalizing constant. In particular, for  $H = \frac{1}{2}$  we have  $Mf(x) = f(x)$ , and for  $0 < H < \frac{1}{2}$  we have

$$Mf(x) = \beta_H \int_{\mathbb{R}} \frac{f(x - s) - f(x)}{|s|^{\frac{3}{2}-H}} ds.$$

As an example let us recall  $M1_{[a,b]}(x)$ , i.e.,  $Mf$  when  $f$  is the indicator function of an interval  $[a, b]$  with  $a < b$ . By Elliott-Van der Hoek [8],  $M1_{[a,b]}(x)$  can be calculated explicitly as

$$(2.5) \quad M1_{[a,b]}(x) = \frac{\sqrt{\Gamma(2H+1)\sin(\pi H)}}{2\Gamma(H+\frac{1}{2})\cos(\frac{\pi}{2}(H+\frac{1}{2}))} \left( \frac{b-x}{|b-x|^{\frac{3}{2}-H}} - \frac{a-x}{|a-x|^{\frac{3}{2}-H}} \right).$$

By using the operator  $M$  we can give the relation between fractional and classical white noise (see Chapter 4 in Biagini *et al* [2])

$$W_t^{(H)} = MW_t,$$

which leads to

$$\int_0^T u_t dB_t^H = \int_{\mathbb{R}} M(u1_{[0,T]})_t \delta B_t,$$

where  $u$  is an adapted process and  $\int_{\mathbb{R}} v_t \delta B_t$  denotes the Skorohod integral with respect to Brownian motion  $B$  defined by

$$\int_{\mathbb{R}} v_t \delta B_t := \int_{\mathbb{R}} v_t \diamond W_t dt.$$

Let  $D_t^{(H)}$  denotes the Hida-Malliavin derivative with respect to  $B^H$ . In the classical case ( $H = 1/2$ ) we use the notation  $D_t$  for the corresponding Hida-Malliavin derivative (for further details, see Nualart [23] and Biagini *et al* [2]). We have

$$D_t F = MD_t^{(H)} F$$

and

$$(2.6) \quad E \left[ F \int_0^T u_s dB_s^H \right] = E \left[ \int_{\mathbb{R}} (Mu1_{[0,T]})_s (MD_s^{(H)} F) ds \right]$$

for  $F \in L^2(P)$ .

## 3. SOME BASIC ESTIMATES

In this section we will establish some technical estimates associated with fractional Brownian motion with  $0 < H < \frac{1}{2}$ . For simplicity throughout this paper we let  $C$  stand for a positive constant depending only on the subscripts and its value may be different in different appearance, and this assumption is also adaptable to  $c$ .

**Lemma 3.1.** *For all  $t, s \in [0, T]$ ,  $t \geq s$  and  $0 < H < 1$  we have*

$$(3.1) \quad \frac{1}{2}(2 - 2^H)s^{2H}(t - s)^{2H} \leq t^{2H}s^{2H} - \mu^2 \leq 2s^{2H}(t - s)^{2H},$$

where  $\mu = E(B_t^H B_s^H)$ .

By the local nondeterminacy of fBm we can prove the lemma. Here, we shall use an elementary method to prove it. We shall use the following inequalities:

$$(3.2) \quad (1 + x)^\alpha \leq 1 + (2^\alpha - 1)x^\alpha$$

$$(3.3) \quad (2 - 2^\alpha)x^\alpha(1 - x)^\alpha \leq (1 - x)^\alpha - (1 - x^\alpha) \leq x^\alpha(1 - x)^\alpha$$

with  $0 \leq x, \alpha \leq 1$ . The inequality (3.2) is a calculus exercise, and it is stronger than the well known (Bernoulli) inequality

$$(1 + x)^\alpha \leq 1 + \alpha x^\alpha \leq 1 + x^\alpha,$$

because  $2^\alpha - 1 \leq \alpha$  for all  $0 \leq \alpha \leq 1$ . The inequalities (3.3) are the improvement of the classical inequality

$$1 - x^\alpha \leq (1 - x)^\alpha.$$

The right inequality in (3.3) follows from the fact

$$(1 - x)^\alpha(1 - x^\alpha) \leq 1 - x^\alpha.$$

For the left inequality in (3.3), by (3.2) we have

$$1 = (1 - x + x)^\alpha \leq (1 - x)^\alpha \vee x^\alpha + (2^\alpha - 1)[(1 - x)^\alpha \wedge x^\alpha]$$

for  $0 \leq x \leq 1$ , where  $x \vee y = \max\{x, y\}$  and  $x \wedge y = \min\{x, y\}$ , which deduces

$$\begin{aligned} (1 - x)^\alpha - (1 - x^\alpha) &\geq (2 - 2^\alpha)(1 - x)^\alpha \wedge x^\alpha \\ &\geq (2 - 2^\alpha)(1 - x)^\alpha x^\alpha. \end{aligned}$$

*Proof of (3.1).* Take  $s = xt, 0 \leq x \leq 1$ . Then we can rewrite  $\rho_{r,s} := t^{2H}s^{2H} - \mu^2$  as

$$\begin{aligned} \rho_{r,s} &= t^{4H} \left\{ x^{2H} - \frac{1}{4} [1 + x^{2H} - (1 - x)^{2H}]^2 \right\} \\ &\equiv t^{4H} G(x). \end{aligned}$$

In order to show the lemma we claim that

$$(3.4) \quad \frac{1}{2}(2 - 2^H)x^{2H}(1 - x)^{2H} \leq G(x) \leq 2x^{2H}(1 - x)^{2H}$$

for all  $x \in [0, 1]$ . We have

$$\begin{aligned}
G(x) &= x^{2H} - \frac{1}{4} [1 + x^{2H} - (1 - x)^{2H}]^2 \\
&= \frac{1}{4} \{2x^H - (1 + x^{2H} - (1 - x)^{2H})\} \{2x^H + (1 + x^{2H} - (1 - x)^{2H})\} \\
&= \frac{1}{4} \{(1 - x)^{2H} - (1 - x^H)^2\} \{2x^H + x^{2H} + 1 - (1 - x)^{2H}\} \\
&= \frac{1}{4} \{(1 - x)^H - (1 - x^H)\} \{(1 - x)^H + 1 - x^H\} \{2x^H + x^{2H} + 1 - (1 - x)^{2H}\}.
\end{aligned}$$

Thus, (3.4) follows from (3.3) and the facts

$$\begin{aligned}
(1 - x)^H &\leq (1 - x)^H + (1 - x^H) \leq 2(1 - x)^H, \\
2x^H &\leq 2x^H + x^{2H} + 1 - (1 - x)^{2H} \leq 4x^H.
\end{aligned}$$

This completes the proof.  $\square$

**Lemma 3.2.** *For all  $t, s \in [0, T]$ ,  $t \geq s$  and  $0 < H < \frac{1}{2}$  we have*

$$(3.5) \quad \frac{1}{2}(t - s)^{2H} \leq t^{2H} - \mu \leq (t - s)^{2H},$$

and

$$(3.6) \quad \frac{1}{2}(2 - 2^H)\left(\frac{s}{t}\right)^{2H}(t - s)^{2H} \leq s^{2H} - \mu \leq \frac{1}{2}\left(\frac{s}{t}\right)^{2H}(t - s)^{2H},$$

where  $\mu = E(B_t^H B_s^H)$ .

*Proof.* The inequalities (3.5) follow from

$$\begin{aligned}
t^{2H} - \mu &= t^{2H} - \frac{1}{2}(t^{2H} + s^{2H} - (t - s)^{2H}) \\
&= \frac{1}{2}(t^{2H} - s^{2H}) + \frac{1}{2}(t - s)^{2H}.
\end{aligned}$$

In order to show that (3.6), we have

$$\begin{aligned}
s^{2H} - \mu &= s^{2H} - \frac{1}{2}(t^{2H} + s^{2H} - (t - s)^{2H}) \\
&= \frac{1}{2}t^{2H} \left\{ \left(1 - \frac{s}{t}\right)^{2H} - \left(1 - \left(\frac{s}{t}\right)^{2H}\right) \right\}.
\end{aligned}$$

Thus, the inequalities (3.6) follow from (3.3). This completes the proof.  $\square$

**Lemma 3.3.** *For  $0 < H < \frac{1}{2}$  we have*

$$(3.7) \quad |E[(B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H)]| \leq C_H \frac{(t - s)^{2H}(t' - s')^{2H}}{(s - t')^{2H}}$$

for all  $0 < s' < t' < s < t$ .

Moreover, the estimate (3.7) holds also for all  $0 < s' < s < t' < t$ . In fact we have

$$\begin{aligned}
(t' - s)^{4H} &= (t' - s)^{2H} (t' - s)^{2H} \leq (t - s)^{2H} (t' - s')^{2H}, \\
(t - t')^{2H} (t' - s)^{2H} &\leq (t - s)^{2H} (t' - s')^{2H}, \\
(s - s')^{2H} (t' - s)^{2H} &\leq (t' - s')^{2H} (t - s)^{2H}, \\
(t - s')^{2H} &= \{(t - s) + (s - s')\}^{2H} \leq (t - s)^{2H} + (s - s')^{2H} \\
&= \frac{(t - s)^{2H} (t' - s)^{2H} + (s - s')^{2H} (t' - s)^{2H}}{(t' - s)^{2H}} \\
&\leq 2 \frac{(t - s)^{2H} (t' - s')^{2H}}{(t' - s)^{2H}},
\end{aligned}$$

which gives

$$\begin{aligned}
|E[(B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H)]| &= \frac{1}{2} \{|t - s'|^{2H} + |s - t'|^{2H} - |t - t'|^{2H} - |s - s'|^{2H}\} \\
&\leq 3 \frac{(t - s)^{2H} (t' - s')^{2H}}{(t' - s)^{2H}}.
\end{aligned}$$

*Proof of (3.7).* For  $0 < s' < t' < s < t \leq T$  we define the function  $x \mapsto G_{s,t}(x)$  on  $[s', t']$  by

$$G_{s,t}(x) = (s - x)^{2H} - (t - x)^{2H}.$$

Thanks to mean value theorem, we see that there are  $\xi \in (s', t')$  and  $\eta \in (s, t)$  such that

$$\begin{aligned}
2E[(B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H)] &= G_{s,t}(t') - G_{s,t}(s') \\
&= 2H(t' - s')[(t - \xi)^{2H-1} - (s - \xi)^{2H-1}] \\
&= 2H(2H - 1)(t' - s')(t - s)(\eta - \xi)^{2H-2} \leq 0,
\end{aligned}$$

which gives

$$(3.8) \quad |E[(B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H)]| \leq \frac{(t' - s')(t - s)}{(s - t')^{2-2H}}.$$

On the other hand, noting that

$$\frac{|E[(B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H)]|}{(t - s)^H (t' - s')^H} \leq 1,$$

we see that

$$\frac{|E[(B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H)]|}{(t - s)^H (t' - s')^H} \leq \left( \frac{|E[(B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H)]|}{(t - s)^H (t' - s')^H} \right)^\alpha$$

for all  $\alpha \in [0, 1]$ . Combining this with (3.8), we get

$$|E[(B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H)]| \leq \frac{(t - s)^{(1-\alpha)H+\alpha} (t' - s')^{(1-\alpha)H+\alpha}}{(s - t')^{\alpha(2-2H)}},$$

and the lemma follows by taking  $\alpha = H/(1 - H)$ .  $\square$



**Lemma 3.4.** For  $0 < H < \frac{1}{2}$  we have

$$\begin{aligned} |E [B_t^H (B_t^H - B_s^H)]| &\leq (t - s)^{2H}, \\ |E [B_t^H (B_s^H - B_r^H)]| &\leq (s - r)^{2H}, \\ |E [B_r^H (B_t^H - B_s^H)]| &\leq (t - s)^{2H} \end{aligned}$$

for all  $t > s > r > 0$ .

Let  $\varphi(x, y)$  be the density function of  $(B_s^H, B_r^H)$  ( $s > r > 0$ ). That is

$$\varphi(x, y) = \frac{1}{2\pi\rho} \exp \left\{ -\frac{1}{2\rho^2} (r^{2H}x^2 - 2\mu xy + s^{2H}y^2) \right\},$$

where  $\mu = E(B_s^H B_r^H)$  and  $\rho^2 = r^{2H}s^{2H} - \mu^2$ .

**Lemma 3.5.** Let  $f \in C^1(\mathbb{R})$  admit compact support. Then we have

$$|E [f'(B_s^H) f'(B_r^H)]| \leq \frac{C_H s^H}{r^H (s - r)^{2H}} (E [|f(B_s^H)|^2] E [|f(B_r^H)|^2])^{1/2}$$

for all  $s > r > 0$  and  $0 < H < \frac{1}{2}$ .

*Proof.* Elementary calculation shows that

$$\begin{aligned} &\int_{\mathbb{R}^2} f^2(y) (x - \frac{\mu}{r^{2H}} y)^2 \varphi(x, y) dx dy \\ &= \frac{\rho^2}{r^{2H}} \int_{\mathbb{R}} f^2(y) \frac{1}{\sqrt{2\pi r^H}} e^{-\frac{y^2}{2r^{2H}}} dy = \frac{\rho^2}{r^{2H}} E [|f(B_r^H)|^2], \end{aligned}$$

which implies that

$$\begin{aligned} &\frac{1}{\rho^4} \int_{\mathbb{R}^2} |f(x) f(y) (s^{2H} y - \mu x) (r^{2H} x - \mu y)| \varphi(x, y) dx dy \\ &\leq \frac{r^H s^H}{\rho^2} (E [|f(B_s^H)|^2] E [|f(B_r^H)|^2])^{1/2} \\ &\leq \frac{C_H s^H}{r^H (s - r)^{2H}} (E [|f(B_s^H)|^2] E [|f(B_r^H)|^2])^{1/2} \end{aligned}$$

by Lemma 3.1. It follows that

$$\begin{aligned} |E [f'(B_s^H) f'(B_r^H)]| &= \left| \int_{\mathbb{R}^2} f(x) f(y) \frac{\partial^2}{\partial x \partial y} \varphi(x, y) dx dy \right| \\ &= \left| \int_{\mathbb{R}^2} f(x) f(y) \left\{ \frac{1}{\rho^4} (s^{2H} y - \mu x) (r^{2H} x - \mu y) + \frac{\mu}{\rho^2} \right\} \varphi(x, y) dx dy \right| \\ &\leq \frac{C_H s^H}{r^H (s - r)^{2H}} (E [|f(B_s^H)|^2] E [|f(B_r^H)|^2])^{1/2}. \end{aligned}$$

This completes the proof.  $\square$

**Lemma 3.6.** Let  $f \in C^2(\mathbb{R})$  admit compact support. Then we have

$$|E [f''(B_s^H) f(B_r^H)]| \leq \frac{C_H}{(s - r)^{2H}} (E [|f(B_s^H)|^2] E [|f(B_r^H)|^2])^{1/2}$$

for all  $s > r > 0$  and  $0 < H < \frac{1}{2}$ .

*Proof.* A straightforward calculation shows that

$$\int_{\mathbb{R}^2} f^2(y) \left(x - \frac{\mu}{r^{2H}} y\right)^4 \varphi(x, y) dx dy = \frac{3\rho^4}{r^{4H}} \int_{\mathbb{R}} f^2(y) \frac{1}{\sqrt{2\pi} r^H} e^{-\frac{y^2}{2r^{2H}}} dy,$$

which deduces

$$\begin{aligned} \frac{1}{\rho^4} \int_{\mathbb{R}^2} f(x) f(y) (r^{2H} x - \mu y)^2 \varphi(x, y) dx dy \\ \leq \frac{C_H}{(s-r)^{2H}} \sqrt{E[|f(B_s^H)|^2] E[|f(B_r^H)|^2]} \end{aligned}$$

by Cauchy's inequality and Lemma 3.1. It follows that

$$\begin{aligned} |E[f''(B_s^H) f(B_r^H)]| &= \left| \int_{\mathbb{R}^2} f(x) f(y) \frac{\partial^2}{\partial x^2} \varphi(x, y) dx dy \right| \\ &= \left| \int_{\mathbb{R}^2} f(x) f(y) \left\{ \frac{1}{\rho^4} (r^{2H} x - \mu y)^2 - \frac{r^{2H}}{\rho^2} \right\} \varphi(x, y) dx dy \right| \\ &\leq \frac{C_H}{(s-r)^{2H}} (E[|f(B_s^H)|^2] E[|f(B_r^H)|^2])^{1/2}. \end{aligned}$$

This completes the proof.  $\square$

#### 4. EXISTENCE OF THE GENERALIZED QUADRATIC COVARIATION

In this section, for  $0 < H < \frac{1}{2}$  we study the existence of the *generalized quadratic covariation*. Denote

$$J_\varepsilon(f, t) := \frac{1}{\varepsilon^{2H}} \int_0^t \{f(B_{s+\varepsilon}^H) - f(B_s^H)\} (B_{s+\varepsilon}^H - B_s^H) ds^{2H}$$

for  $\varepsilon > 0$  and  $t \geq 0$ . Recall that the *generalized quadratic covariation*  $[f(B^H), B^H]_t^{(W)}$  is defined as

$$(4.1) \quad [f(B^H), B^H]_t^{(W)} := \lim_{\varepsilon \downarrow 0} J_\varepsilon(f, t),$$

provided the limit exists uniformly in probability. Clearly, we have (see, for example, Klein and Giné [16])

$$(4.2) \quad [B^H, B^H]_t^{(W)} = t^{2H}$$

for all  $t \geq 0$ . In fact, one can easily prove that

$$\begin{aligned} E \left| \frac{1}{\varepsilon^{2H}} \int_0^t (B_{s+\varepsilon}^H - B_s^H)^2 ds - t^{2H} \right|^2 \\ = \frac{1}{\varepsilon^{4H}} \int_0^t \int_0^t E[(B_{r+\varepsilon}^H - B_r^H)^2 (B_{s+\varepsilon}^H - B_s^H)^2] ds^{2H} dr^{2H} - t^{4H} \\ \longrightarrow 0 \end{aligned}$$

for  $t \geq 0$ , as  $\varepsilon \downarrow 0$ .

Consider the decomposition

$$\begin{aligned}
 (4.3) \quad & \frac{1}{\varepsilon^{2H}} \int_0^t \{f(B_{s+\varepsilon}^H) - f(B_s^H)\} (B_{s+\varepsilon}^H - B_s^H) ds^{2H} \\
 &= \frac{1}{\varepsilon^{2H}} \int_0^t f(B_{s+\varepsilon}^H) (B_{s+\varepsilon}^H - B_s^H) ds^{2H} - \frac{1}{\varepsilon^{2H}} \int_0^t f(B_s^H) (B_{s+\varepsilon}^H - B_s^H) ds^{2H} \\
 &\equiv I_\varepsilon^+(f, t) - I_\varepsilon^-(f, t),
 \end{aligned}$$

and define the set  $\mathcal{H} = \{f : \text{measurable functions on } \mathbb{R} \text{ such that } \|f\|_{\mathcal{H}} < \infty\}$ , where

$$\|f\|_{\mathcal{H}} := \sqrt{\int_0^T \int_{\mathbb{R}} |f(x)|^2 e^{-\frac{x^2}{2s^{2H}}} \frac{dx ds}{\sqrt{2\pi}s^{1-H}}} + \sqrt{\int_0^T \int_{\mathbb{R}} |f(x)|^2 e^{-\frac{x^2}{2s^{2H}}} \frac{dx ds}{\sqrt{2\pi}(T-s)^{1-H}}}.$$

Then,  $\mathcal{H}$  is a Banach space and the set  $\mathcal{E}$  of elementary functions of the form

$$f_\Delta(x) = \sum_i f_i 1_{(x_{i-1}, x_i]}(x)$$

is dense in  $\mathcal{H}$ , where  $\{x_i, 0 \leq i \leq l\}$  is a finite sequence of real numbers such that  $x_i < x_{i+1}$ . Moreover,  $\mathcal{H}$  contains the sets  $\mathcal{H}_\gamma$ ,  $\gamma > 2$ , of measurable functions  $f$  such that

$$\int_0^T \int_{\mathbb{R}} |f(x)|^\gamma e^{-\frac{x^2}{2s^{2H}}} \frac{dx ds}{\sqrt{2\pi}s^{1-H}} < \infty.$$

Our main object of this section is to explain and prove the following theorem.

**Theorem 4.1.** *Let  $0 < H < \frac{1}{2}$  and  $f \in \mathcal{H}$ . Then the generalized quadratic covariation  $[f(B^H), B^H]^{(W)}$  exists and*

$$(4.4) \quad E \left| [f(B^H), B^H]_t^{(W)} \right|^2 \leq C_H \|f\|_{\mathcal{H}}^2.$$

We split the proof into several lemmas, and for simplicity throughout this paper we let  $T = 1$ .

**Lemma 4.1.** *Let  $0 < H < \frac{1}{2}$  and let  $f$  be an infinitely differentiable function with compact support. We then have*

$$(4.5) \quad E \left| I_\varepsilon^-(f, t) \right|^2 \leq C_H \|f\|_{\mathcal{H}}^2,$$

$$(4.6) \quad E \left| I_\varepsilon^+(f, t) \right|^2 \leq C_H \|f\|_{\mathcal{H}}^2$$

for all  $0 < \varepsilon \leq 1$ .

*Proof.* We need only to obtain the first estimate. It follows from (2.6) that

$$\begin{aligned}
& E [f(B_s^H) f(B_r^H) (B_{s+\varepsilon}^H - B_s^H) (B_{r+\varepsilon}^H - B_r^H)] \\
&= E \left[ f(B_s^H) f(B_r^H) (B_{s+\varepsilon}^H - B_s^H) \int_r^{r+\varepsilon} dB_l^H \right] \\
&= E \int_{\mathbb{R}} M1_{[r, r+\varepsilon]}(l) M D_l^{(H)} f(B_s^H) f(B_r^H) (B_{s+\varepsilon}^H - B_s^H) dl \\
&= \int_{\mathbb{R}} M1_{[r, r+\varepsilon]}(l) M1_{[0, s]}(l) E [f'(B_s^H) f(B_r^H) (B_{s+\varepsilon}^H - B_s^H)] dl \\
&\quad + \int_{\mathbb{R}} M1_{[r, r+\varepsilon]}(l) M1_{[0, r]}(l) E [f(B_s^H) f'(B_r^H) (B_{s+\varepsilon}^H - B_s^H)] dl \\
&\quad + \int_{\mathbb{R}} M1_{[r, r+\varepsilon]}(l) M1_{[s, s+\varepsilon]}(l) E [f(B_s^H) f(B_r^H)] dl \\
&= E [B_s^H (B_{r+\varepsilon}^H - B_r^H)] E [f'(B_s^H) f(B_r^H) (B_{s+\varepsilon}^H - B_s^H)] \\
&\quad + E [B_r^H (B_{r+\varepsilon}^H - B_r^H)] E [f(B_s^H) f'(B_r^H) (B_{s+\varepsilon}^H - B_s^H)] \\
&\quad + E [(B_{r+\varepsilon}^H - B_r^H) (B_{s+\varepsilon}^H - B_s^H)] E [f(B_s^H) f(B_r^H)] \\
&\equiv \Psi_\varepsilon(s, r, 1) + \Psi_\varepsilon(s, r, 2) + \Psi_\varepsilon(s, r, 3).
\end{aligned}$$

In order to end the proof we claim now that

$$(4.7) \quad \frac{1}{\varepsilon^{4H}} \left| \int_0^t \int_0^t \Psi_\varepsilon(s, r, k) ds^{2H} dr^{2H} \right| \leq C_H \|f\|_{\mathcal{H}}^2, \quad k = 1, 2, 3,$$

for all  $\varepsilon > 0$  small enough. Some elementary calculus can show that, for all  $0 < \varepsilon \leq 1$

$$\begin{aligned}
& \int_\varepsilon^1 E [|f(B_s^H)|^2] s^{2H-1} ds \int_0^{s-\varepsilon} \frac{dr}{r^{1-2H}(s-\varepsilon-r)^{2H}} \\
&= \int_\varepsilon^1 E [|f(B_s^H)|^2] s^{2H-1} ds \int_0^{s-\varepsilon} \frac{dr}{r^{1-2H}(s-\varepsilon-r)^{2H}} \\
&= \int_\varepsilon^1 s^{2H-1} E [|f(B_s^H)|^2] ds \left( \int_0^1 \frac{dr}{x^{1-2H}(1-x)^{2H}} dx \right), \\
& \int_\varepsilon^1 E [|f(B_s^H)|^2] s^{2H-1} ds \int_{s-\varepsilon}^s \frac{dr}{r^{1-2H}(r+\varepsilon-s)^{2H}} \\
&\leq \int_\varepsilon^1 E [|f(B_s^H)|^2] ds \int_{s-\varepsilon}^s \frac{dr}{r^{2-4H}(r+\varepsilon-s)^{2H}} \\
&= \int_\varepsilon^1 E [|f(B_s^H)|^2] ds \int_1^{\frac{s}{s-\varepsilon}} \frac{dx}{x^{2-4H}(x-1)^{2H}} \\
&\leq \int_0^1 E [|f(B_s^H)|^2] ds \left( \int_1^{+\infty} \frac{dx}{x^{2-4H}(x-1)^{2H}} \right),
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^\varepsilon E [|f(B_s^H)|^2] s^{2H-1} ds \int_0^s \frac{r^{2H-1} dr}{(r + \varepsilon - s)^{2H}} \\
&= \int_0^\varepsilon E [|f(B_s^H)|^2] s^{2H-1} ds \int_0^{\frac{s}{\varepsilon-s}} \frac{x^{2H-1} dx}{(1+x)^{2H}} \\
&\leq \int_0^\varepsilon E [|f(B_s^H)|^2] ds \frac{s^{3H-1}}{(\varepsilon-s)^H} \left( \int_0^\infty \frac{x^{H-1} dx}{(1+x)^{2H}} \right) \\
(4.8) \quad &\leq C_H \int_0^1 \int_{\mathbb{R}} |f(x)|^2 e^{-\frac{x^2}{2s^{2H}}} \frac{s^{2H-1} dx ds}{\sqrt{2\pi}(1-s)^H},
\end{aligned}$$

where the estimate (4.8) follows from the monotonicity of the function

$$\varepsilon \mapsto \int_0^\varepsilon \frac{s^{2H-1}}{(\varepsilon-s)^H} e^{-\frac{x^2}{2s^{2H}}} ds$$

with  $\varepsilon \in [0, 1]$ . It follows that

$$\begin{aligned}
& \frac{1}{\varepsilon^{4H}} \left| \int_0^1 \int_0^1 \Psi_\varepsilon(s, r, 3) ds^{2H} dr^{2H} \right| \\
&\leq \frac{H}{\varepsilon^{4H}} \int_0^1 \int_0^1 |E [(B_{r+\varepsilon}^H - B_r^H)(B_{s+\varepsilon}^H - B_s^H)]| \\
&\quad \cdot \{E [f^2(B_s^H)] + E [f^2(B_r^H)]\} (sr)^{2H-1} ds dr \\
&= \frac{H}{\varepsilon^{4H}} \int_0^1 \int_0^1 |E [(B_{r+\varepsilon}^H - B_r^H)(B_{s+\varepsilon}^H - B_s^H)]| E [f^2(B_s^H)] (sr)^{2H-1} ds dr \\
&\leq H \int_\varepsilon^1 E [|f(B_s^H)|^2] s^{2H-1} ds \int_0^{s-\varepsilon} \frac{dr}{r^{1-2H}(s-\varepsilon-r)^{2H}} \\
&\quad + H \int_\varepsilon^1 E [|f(B_s^H)|^2] s^{2H-1} ds \int_{s-\varepsilon}^s \frac{dr}{r^{1-2H}(r+\varepsilon-s)^{2H}} \\
&\quad + H \int_0^\varepsilon E [|f(B_s^H)|^2] s^{2H-1} ds \int_0^s \frac{r^{2H-1} dr}{(r+\varepsilon-s)^{2H}} \\
&\leq C_H \|f\|_{\mathcal{H}}^2
\end{aligned}$$

for all  $0 < \varepsilon \leq 1$ .

Now, let us obtain the estimate (4.7) for  $k = 1$ . By (2.6) we see that

$$\begin{aligned}
\Psi_\varepsilon(s, r, 1) &= E [B_s^H (B_{r+\varepsilon}^H - B_r^H)] E [f'(B_s^H) f(B_r^H) (B_{s+\varepsilon}^H - B_s^H)] \\
&= E [B_s^H (B_{r+\varepsilon}^H - B_r^H)] E [B_s^H (B_{s+\varepsilon}^H - B_s^H)] E [f''(B_s^H) f(B_r^H)] \\
&\quad + E [B_s^H (B_{r+\varepsilon}^H - B_r^H)] E [B_r^H (B_{s+\varepsilon}^H - B_s^H)] E [f'(B_s^H) f'(B_r^H)] \\
&\equiv \Psi_\varepsilon(s, r, 1, 1) + \Psi_\varepsilon(s, r, 1, 2).
\end{aligned}$$

Together Lemma 3.5, Lemma 3.6, Lemma 3.4 and the fact

$$\begin{aligned}
(4.9) \quad E [f^2(B_r^H)] &= \int_{\mathbb{R}} f^2(x) \frac{1}{\sqrt{2\pi} r^H} e^{-\frac{x^2}{2r^{2H}}} dx \\
&\leq \frac{s^H}{r^H} \int_{\mathbb{R}} f^2(x) \frac{1}{\sqrt{2\pi} s^H} e^{-\frac{x^2}{2s^{2H}}} dx = \frac{s^H}{r^H} E [f^2(B_s^H)]
\end{aligned}$$

with  $s \geq r > 0$  lead to

$$\begin{aligned}
\frac{1}{\varepsilon^{4H}} \left| \int_0^t \int_0^t \Psi_\varepsilon(s, r, 1, 1) ds^{2H} dr^{2H} \right| &\leq \int_0^t \int_0^t |E[f''(B_s^H)f(B_r^H)]| ds^{2H} dr^{2H} \\
&\leq C_H \int_0^t \int_0^s \frac{1}{(s-r)^{2H}} E|f(B_s^H)f(B_r^H)| ds^{2H} dr^{2H} \\
&\leq C_H \int_0^t E[f^2(B_s^H)] ds^{2H} \int_0^s \frac{s^{H/2}}{(s-r)^{2H} r^{H/2}} dr^{2H} \\
&\leq C_H \|f\|_{\mathcal{H}}^2,
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\varepsilon^{4H}} \left| \int_0^t \int_0^t \Psi_\varepsilon(s, r, 1, 2) ds^{2H} dr^{2H} \right| &\leq \int_0^t \int_0^t |E[f'(B_s^H)f'(B_r^H)]| ds^{2H} dr^{2H} \\
&\leq C_H \int_0^t \int_0^s \frac{s^H}{r^H(s-r)^{2H}} |E[f(B_s^H)f(B_r^H)]| ds^{2H} dr^{2H} \\
&\leq C_H \|f\|_{\mathcal{H}}^2
\end{aligned}$$

for all  $\varepsilon > 0$  and  $t \geq 0$ . Thus, we get

$$\frac{1}{\varepsilon^{4H}} \left| \int_0^t \int_0^t \Psi_\varepsilon(s, r, 1) ds^{2H} dr^{2H} \right| \leq C_H \|f\|_{\mathcal{H}}^2.$$

Similarly, we can also obtain the estimate (4.7) for  $k = 2$ , and the lemma follows.  $\square$

Recently, Gradinaru-Nourdin [12] introduced the following perfect result:

**Theorem A** (Theorem 2.1 in Gradinaru–Nourdin [12]). *Assume that  $H \in (0, 1)$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function satisfying*

$$(4.10) \quad |f(x) - f(y)| \leq C|x - y|^a(1 + x^2 + y^2)^b, \quad (C > 0, 0 < a \leq 1, b > 0),$$

for all  $x, y \in \mathbb{R}$ , and let  $\{Y_t : t \geq 0\}$  be a continuous stochastic process. Then, as  $\varepsilon \rightarrow 0$ ,

$$(4.11) \quad \int_0^t Y_s f\left(\frac{B_{s+\varepsilon}^H - B_s^H}{\varepsilon^H}\right) ds \longrightarrow E[f(N)] \int_0^t Y_s ds,$$

almost surely, uniformly in  $t$  on each compact interval, where  $N$  is a standard Gaussian random variable.

According to the theorem above we get the next lemma.

**Lemma 4.2.** *Let  $0 < H < 1$  and  $f \in C(\mathbb{R})$ . We then have*

$$(4.12) \quad \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2H}} \int_0^t f(B_s^H)(B_{s+\varepsilon}^H - B_s^H)^2 ds^{2H} = \int_0^t f(B_s^H) ds^{2H}$$

almost surely, for all  $t \geq 0$ .

As a direct consequence of Lemma 4.2, for  $f \in C^1(\mathbb{R})$  we have

$$(4.13) \quad [f(B^H), B^H]_t^{(W)} = 2H \int_0^t f'(B_s^H) s^{2H-1} ds$$

for all  $0 < H < 1$ . In fact, the Hölder continuity of fBm  $B^H$  yields

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2H}} \int_0^t o(B_{s+\varepsilon}^H - B_s^H)(B_{s+\varepsilon}^H - B_s^H)^2 ds^{2H} = 0$$

almost surely. It follows that

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2H}} \int_0^t \{f(B_{s+\varepsilon}^H) - f(B_s^H)\} (B_{s+\varepsilon}^H - B_s^H) ds^{2H} \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2H}} \int_0^t f'(B_s^H)(B_{s+\varepsilon}^H - B_s^H)^2 ds^{2H} = \int_0^t f'(B_s^H) ds^{2H} \end{aligned}$$

almost surely.

Now we can show our main result.

*Proof of Theorem 4.1.* Given  $f \in \mathcal{H}$ . If  $f \in C^1(\mathbb{R})$ , then the theorem follows from the identity (4.13) and the follows estimate:

$$\begin{aligned} E \left( \int_0^t f'(B_s^H) s^{2H-1} ds \right)^2 &= \int_0^t \int_0^t E [f'(B_s^H) f'(B_r^H)] (sr)^{2H-1} ds dr \\ &\leq C_H \int_0^t \int_0^s \frac{s^{\frac{7H}{2}-1}}{r^{1-\frac{H}{2}}(s-r)^{2H}} E [f^2(B_s^H)] ds dr \\ &\leq C_H \int_0^t s^{2H-1} E [f^2(B_s^H)] ds \leq C_H \|f\|_{\mathcal{H}}^2 \end{aligned}$$

by Lemma 3.5 and (4.9). Let now  $f \notin C_0^\infty(\mathbb{R})$ .

Consider the function  $\zeta$  on  $\mathbb{R}$  by

$$(4.14) \quad \zeta(x) := \begin{cases} ce^{\frac{1}{(x-1)^{2-1}}}, & x \in (0, 2), \\ 0, & \text{otherwise,} \end{cases}$$

where  $c$  is a normalizing constant such that  $\int_{\mathbb{R}} \zeta(x) dx = 1$ . Define the so-called mollifiers

$$(4.15) \quad \zeta_n(x) := n\zeta(nx), \quad n = 1, 2, \dots$$

and the sequence of smooth functions

$$(4.16) \quad f_n(x) = \int_{\mathbb{R}} f(x-y)\zeta_n(y)dy = \int_0^2 f(x-\frac{y}{n})\zeta(y)dy, \quad n = 1, 2, \dots$$

for all  $x \in \mathbb{R}$ . Then  $\{f_n\} \subset C^\infty(\mathbb{R}) \cap \mathcal{H}$  and  $f_n$  converges to  $f$  in  $\mathcal{H}$ , as  $n$  tends to infinity.

On the other hand, by Lemma 4.1 we have

$$\begin{aligned} P(|J_{\varepsilon_1}(f, t) - J_{\varepsilon_2}(f, t)| \geq \delta) &\leq P\left(|J_{\varepsilon_1}(f - f_n, t)| \geq \frac{\delta}{3}\right) + P\left(|J_{\varepsilon_2}(f - f_n, t)| \geq \frac{\delta}{3}\right) \\ &\quad + P\left(|J_{\varepsilon_1}(f_n, t) - J_{\varepsilon_2}(f_n, t)| \geq \frac{\delta}{3}\right) \\ &\leq \frac{C_H}{\delta^2} \|f - f_n\|_{\mathcal{H}}^2 + P\left(|J_{\varepsilon_1}(f_n, t) - J_{\varepsilon_2}(f_n, t)| \geq \frac{\delta}{3}\right) \end{aligned}$$

for all  $n$  and  $\delta, \varepsilon_1, \varepsilon_2 > 0$ . Combining this with

$$\lim_{\varepsilon \downarrow 0} J_\varepsilon(f_n, t) = [f_n(B^H), B^H]_t^{(W)} = 2H \int_0^t f'_n(B_s^H) s^{2H-1} ds, \quad n \geq 1$$

in probability, we show that the generalized quadratic covariation  $[f(B^H), B^H]^{(W)}$  exists for  $f \in \mathcal{H}$ . Thus, the estimate (4.4) follows from Lemma 4.1. This completes the proof.  $\square$

**Corollary 4.1.** *Let  $f, f_1, f_2, \dots \in \mathcal{H}$ . If  $f_n \rightarrow f$  in  $\mathcal{H}$ , as  $n$  tends to infinity, then we have*

$$[f_n(B^H), B^H]_t^{(W)} \longrightarrow [f(B^H), B^H]_t^{(W)}$$

in  $L^2$  as  $n \rightarrow \infty$ .

*Proof.* The convergence follows from

$$E \left| [f_n(B^H), B^H]_t^{(W)} - [f(B^H), B^H]_t^{(W)} \right|^2 \leq C_H \|f_n - f\|_{\mathcal{H}}^2 \rightarrow 0,$$

as  $n$  tends to infinity.  $\square$

By using the above result, we immediately get an extension of Itô formula stated as follows.

**Theorem 4.2.** *Let  $0 < H < \frac{1}{2}$  and let  $f \in \mathcal{H}$  be left continuous. If  $F$  is an absolutely continuous function with the derivative  $F' = f$ , then the following Itô type formula holds:*

$$(4.17) \quad F(B^H) = F(0) + \int_0^t f(B_s^H) dB_s^H + \frac{1}{2} [f(B^H), B^H]_t^{(W)}.$$

Clearly, this is an analogue of Föllmer-Protter-Shiryayev's formula (see Eisenbaum [7], Föllmer *et al* [10], Moret-Nualart [20], Russo-Vallois [26], and the references therein). It is an improvement in terms of the hypothesis on  $f$  and it is also quite interesting itself.

*Proof of Theorem 4.2.* If  $F \in C^2(\mathbb{R})$ , then this is Itô's formula since

$$[f(B^H), B^H]_t^{(W)} = 2H \int_0^t f'(B_s^H) s^{2H-1} ds.$$

For  $F \notin C^2(\mathbb{R})$ , by a localization argument we may assume that the function  $f$  is uniformly bounded. In fact, for any  $k \geq 0$  we may consider the set

$$\Omega_k = \left\{ \sup_{0 \leq t \leq T} |B_t^H| < k \right\}$$

and let  $f^{[k]}$  be a measurable function such that  $f^{[k]} = f$  on  $[-k, k]$  and such that  $f^{[k]}$  vanishes outside. Then  $f^{[k]}$  is uniformly bounded and  $f^{[k]} \in \mathcal{H}$  for every  $k \geq 0$ . Set  $\frac{d}{dx} F^{[k]} = f^{[k]}$  and  $F^{[k]} = F$  on  $[-k, k]$ . If the theorem is true for all uniformly bounded functions on  $\mathcal{H}$ , then we get the desired formula

$$F^{[k]}(B_t^H) = F^{[k]}(0) + \int_0^t f^{[k]}(B_s^H) dB_s^H + \frac{1}{2} [f^{[k]}(B^H), B^H]_t^{(W)}$$

on the set  $\Omega_k$ . Letting  $k$  tend to infinity we deduce the Itô formula (4.17) for all  $f \in \mathcal{H}$  being left continuous and locally bounded.



Let now  $F' = f \in \mathcal{H}$  be uniformly bounded and left continuous. For any positive integer  $n$  we define

$$F_n(x) := \int_{\mathbb{R}} F(x-y)\zeta_n(y)dy, \quad x \in \mathbb{R},$$

where  $\zeta_n$ ,  $n \geq 1$  are the mollifiers defined by (4.15). Then  $F_n \in C^\infty(\mathbb{R})$  for all  $n \geq 1$  and the Itô formula

$$(4.18) \quad F_n(B_t^H) = F_n(0) + \int_0^t f_n(B_s^H)dB_s^H + H \int_0^t f'_n(B_s^H)s^{2H-1}ds$$

holds for all  $n \geq 1$ , where  $f_n = F'_n$ . Moreover using Lebesgue's dominated convergence theorem, one can prove that as  $n \rightarrow \infty$ , for each  $x$ ,

$$F_n(x) \rightarrow F(x), \quad f_n(x) \rightarrow f(x),$$

and  $\{f_n\} \subset \mathcal{H}$ ,  $f_n \rightarrow f$  in  $\mathcal{H}$ , as  $n$  tends to infinity. It follows that

$$2H \int_0^t f'_n(B_s^H)s^{2H-1}ds = [f_n(B^H), B^H]_t^{(W)} \rightarrow [f(B^H), B^H]_t^{(W)}$$

in  $L^2$  by Corollary 4.1, as  $n$  tends to infinity. It follows that

$$\begin{aligned} \int_0^t f_n(B_s^H)dB_s^H &= F_n(B_t^H) - F_n(0) - \frac{1}{2}[f_n(B^H), B^H]_t^{(W)} \\ &\rightarrow F(B_t^H) - F(0) - \frac{1}{2}[f(B^H), B^H]_t^{(W)} \end{aligned}$$

in  $L^2$ , as  $n$  tends to infinity. This completes the proof since the integral is closed in  $L^2$ .  $\square$

## 5. INTEGRATION WITH RESPECT TO THE LOCAL TIME

In this section we assume that  $0 < H < \frac{1}{2}$  and study the integral

$$(5.1) \quad \int_{\mathbb{R}} f(x)\mathcal{L}^H(dx, t),$$

where  $f$  is a deterministic function and

$$\mathcal{L}^H(x, t) = 2H \int_0^t \delta(B_s^H - x)s^{2H-1}ds$$

is the weighted local time of fBm  $B^H$ . Recall that the quadratic covariation  $[f(B), B]$  of Brownian motion  $B$  can be characterized as

$$[f(B), B]_t = - \int_{\mathbb{R}} f(x)\mathcal{L}^B(dx, t),$$

where  $f$  is locally square integrable and  $\mathcal{L}^B(x, t)$  is the local time of Brownian motion. This is called the Bouleau-Yor identity. More works for this can be found in Bouleau-Yor [3], Eisenbaum [7], Föllmer *et al* [10], Feng-Zhao [9], Peskir [24], Rogers-Walsh [25], Yang-Yan [29], and the references therein. However, this is not true for fractional Brownian motion. For  $\frac{1}{2} < H < 1$ , Yan *et al* [28, 27] obtained the following Bouleau-Yor identity:

$$[f(B^H), B^H]_t^{(W)} = - \int_{\mathbb{R}} f(x)\mathcal{L}^H(dx, t).$$

In this section we show that the identity above also holds for  $0 < H < \frac{1}{2}$ .

Take  $F(x) = (x - a)^+ - (x - b)^+$ . Then  $F$  is absolutely continuous with the derivative  $F' = 1_{(a,b]} \in \mathcal{H}$  being left continuous and bounded, and the Itô formula (4.17) yields

$$\begin{aligned} [1_{(a,b]}(B^H), B^H]_t^{(W)} &= 2F(B_t^H) - 2F(0) - 2 \int_0^t 1_{(a,b]}(B_s^H) dB_s^H \\ &= \mathcal{L}^H(a, t) - \mathcal{L}^H(b, t) \end{aligned}$$

for all  $t \in [0, 1]$ . Thus, the linearity property of generalized quadratic covariation deduces the following result.

**Lemma 5.1.** *For any  $f_\Delta(x) = \sum_j f_j 1_{(a_{j-1}, a_j]}(x) \in \mathcal{E}$ , the integral*

$$\int_{\mathbb{R}} f_\Delta(x) \mathcal{L}^H(dx, t) := \sum_j f_j [\mathcal{L}^H(a_j, t) - \mathcal{L}^H(a_{j-1}, t)]$$

*exists and*

$$(5.2) \quad \int_{\mathbb{R}} f_\Delta(x) \mathcal{L}^H(dx, t) = - [f_\Delta(B^H), B^H]_t^{(W)}$$

*for all  $t \in [0, 1]$ .*

Thanks to the density of  $\mathcal{E}$  in  $\mathcal{H}$ , we can then extend the definition of integration with respect to  $x \mapsto \mathcal{L}^H(x, t)$  to the elements of  $\mathcal{H}$  in the following manner:

$$\int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t) := \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_{\Delta, n}(x) \mathcal{L}^H(dx, t)$$

in  $L^2$  for  $f \in \mathcal{H}$  provided  $f_{\Delta, n} \rightarrow f$  in  $\mathcal{H}$ , as  $n$  tends to infinity, where  $\{f_{\Delta, n}\} \subset \mathcal{E}$ . The limit obtained does not depend on the choice of the sequence  $\{f_{\Delta, n}\}$  and represents the integral of  $f$  with respect to  $\mathcal{L}^H$ . Together this and Corollary 4.1 lead to the Bouleau-Yor identity

$$(5.3) \quad [f(B^H), B^H]_t^{(W)} = - \int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t)$$

for all  $t \in [0, 1]$ .

**Corollary 5.1.** *Let  $0 < H < \frac{1}{2}$  and let  $f, f_1, f_2, \dots \in \mathcal{H}$ . If  $f_n \rightarrow f$  in  $\mathcal{H}$ , as  $n$  tends to infinity, we then have*

$$\int_{\mathbb{R}} f_n(x) \mathcal{L}^H(dx, t) \longrightarrow \int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t)$$

*in  $L^2$ , as  $n$  tends to infinity.*

According to Theorem 4.2, we get an analogue of Bouleau-Yor's formula.

**Corollary 5.2.** *Let  $0 < H < \frac{1}{2}$  and let  $f \in \mathcal{H}$  be left continuous. If  $F$  is an absolutely continuous function with the derivative  $F' = f$ , then the following Itô type formula holds:*

$$(5.4) \quad F(B_t^H) = F(0) + \int_0^t f(B_s^H) dB_s^H - \frac{1}{2} \int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t).$$

Recall that if  $F$  is the difference of two convex functions, then  $F$  is an absolutely continuous function with derivative of bounded variation. Thus, the Itô-Tanaka formula

$$\begin{aligned} F(B_t^H) &= F(0) + \int_0^t F'(B_s^H) dB_s^H + \frac{1}{2} \int_{\mathbb{R}} \mathcal{L}^H(x, t) F''(dx) \\ &\equiv F(0) + \int_0^t F'(B_s^H) dB_s^H - \frac{1}{2} \int_{\mathbb{R}} F'(x) \mathcal{L}^H(dx, t) \end{aligned}$$

holds. This is given by Coutin *et al* [4] (see also Hu *et al* [17]).

**Remark 1.** By the proof similar to Lemma 3.1 in Gradinaru–Nourdin [12], one can obtain the following convergence (see also Gradinaru–Nourdin [13]):

$$(5.5) \quad \lim_{n \rightarrow \infty} \sum_{j=1}^n (\Lambda_j)^{2H-1} g(B_{t_j}^H) (B_{t_j}^H - B_{t_{j-1}}^H)^2 = \int_0^t g(B_s^H) s^{2H-1} ds$$

almost surely, where  $\pi_n = \{0 = t_0 < t_1 < \dots < t_n = t\}$  denotes an arbitrary partition of the interval  $[0, t]$  with  $\|\pi_n\| = \sup_j (t_j - t_{j-1}) \rightarrow 0$ ,  $\Lambda_j = \frac{t_j}{t_j - t_{j-1}}$  and  $g \in C(\mathbb{R})$ . Thus, similar to proof of Theorem 4.1 we can show that the convergence

$$2H \lim_{n \rightarrow \infty} \sum_{j=1}^n (\Lambda_j)^{2H-1} \{f(B_{t_j}^H) - f(B_{t_{j-1}}^H)\} (B_{t_j}^H - B_{t_{j-1}}^H) = - \int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t)$$

holds, which deduces

$$2H \lim_{n \rightarrow \infty} \sum_{j=1}^n (\Lambda_j)^{2H-1} \{f(B_{t_j}^H) - f(B_{t_{j-1}}^H)\} (B_{t_j}^H - B_{t_{j-1}}^H) = [f(B^H), B^H]_t^{(W)},$$

where  $f \in \mathcal{H}$  and the limits are uniform in probability.

## 6. THE TIME-DEPENDENT CASE

In this section we consider the time-dependent case. For a measurable function  $f$  on  $\mathbb{R} \times \mathbb{R}_+$  we define the generalized quadratic covariation  $[f(B^H, \cdot), B^H]^{(W)}$  of  $f(B^H, \cdot)$  and  $B^H$  as follows

$$(6.1) \quad [f(B^H, \cdot), B^H]_t^{(W)} := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2H}} \int_0^t \{f(B_{s+\varepsilon}^H, s+\varepsilon) - f(B_s^H, s)\} (B_{s+\varepsilon}^H - B_s^H) ds^{2H}$$

for  $t \in [0, T]$ , provided the limit exists uniformly in probability. We prove the existence of the quadratic covariation.

Consider the set  $\mathcal{H}_*$  of measurable functions  $f$  on  $\mathbb{R} \times \mathbb{R}_+$  such that the function  $t \mapsto f(\cdot, t)$  is continuous and  $\|f\|_{\mathcal{H}_*} < +\infty$ , where

$$\|f\|_{\mathcal{H}_*} = \sqrt{\int_0^T \int_{\mathbb{R}} |f(x, s)|^2 e^{-\frac{x^2}{2s^{2H}}} \frac{dx ds}{\sqrt{2\pi} s^{1-H}}} + \sqrt{\int_0^T \int_{\mathbb{R}} |f(x, s)|^2 e^{-\frac{x^2}{2s^{2H}}} \frac{dx ds}{\sqrt{2\pi} (T-s)^{1-H}}}$$

with  $\varphi_s(x) = \frac{1}{\sqrt{2\pi} s^H} e^{-\frac{x^2}{2s^{2H}}}$ . Then  $\mathcal{H}_*$  is a Banach space and the set  $\mathcal{E}_*$  of elementary functions of the form

$$(6.2) \quad f_{\Delta}(x, t) = \sum_{i,j} f_{ij} 1_{(x_{i-1}, x_i]}(x) 1_{(s_{j-1}, s_j]}(t)$$

is dense in  $\mathcal{H}_*$ , where  $\{x_i, 0 \leq i \leq n\}$  is a finite sequence of real numbers such that  $x_i < x_{i+1}$ ,  $\{s_j, 0 \leq j \leq m\}$  is a subdivision of  $[0, T]$  and  $(f_{ij})$  is a matrix of order  $n \times m$ . Moreover,  $\mathcal{H}_*$  contains the set  $\mathcal{H}_{*,\gamma}$  with  $\gamma > 2$  of measurable functions  $f$  on  $\mathbb{R}$  such that

$$\int_0^T \int_{\mathbb{R}} |f(x, s)|^\gamma e^{-\frac{x^2}{2s^{2H}}} \frac{dx ds}{\sqrt{2\pi} s^{1-H}} < \infty.$$

As a corollary of Theorem A, we have

$$(6.3) \quad \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2H}} \int_0^t s^{2H-1} g(B_s^H, s) (B_{s+\varepsilon}^H - B_s^H)^2 ds = \int_0^t g(B_s^H, s) s^{2H-1} ds$$

almost surely, for all  $t \geq 0$  if  $g$  is continuous. This proves the following identity:

$$(6.4) \quad [f(B^H, \cdot), B^H]_t^{(W)} = 2H \int_0^t \frac{\partial f}{\partial x}(B_s^H, s) s^{2H-1} ds$$

for all  $t \geq 0$ , provided  $f \in C^{1,1}(\mathbb{R} \times \mathbb{R}_+)$ . Thus, similar to proof of Theorem 4.1, one can obtain the next theorem.

**Theorem 6.1.** *Let  $0 < H < \frac{1}{2}$ . If  $f \in \mathcal{H}_*$ , then the generalized quadratic covariation  $[f(B^H, \cdot), B^H]^{(W)}$  exists and*

$$(6.5) \quad E \left| [f(B^H, \cdot), B^H]_t^{(W)} \right|^2 \leq C_H \|f\|_{\mathcal{H}_*}^2$$

for all  $t \in [0, T]$ .

By using the above result, we immediately get an extension of Itô formula stated as follows.

**Theorem 6.2.** *Let  $0 < H < \frac{1}{2}$  and let  $F \in C^{1,1}(\mathbb{R} \times \mathbb{R}_+)$ . Suppose that the function  $\frac{\partial}{\partial x} F = f \in \mathcal{H}_*$ . Then the Itô type formula*

$$F(B_t^H, t) = F(0, 0) + \int_0^t f(B_s^H, s) dB_s^H + \int_0^t \frac{\partial}{\partial t} F(B_s^H, s) ds + \frac{1}{2} [f(B^H, \cdot), B^H]_t^{(W)}$$

holds.

*Proof.* Similar to the proof of Theorem 4.2, we can use smoothing procedure to prove our result. The main different key point is the following approximation:

$$F_n(x, s) := \int \int_{\mathbb{R}^2} F(x - y, s - r) \zeta_n(y) \zeta_n(r) dy dr, \quad n \geq 1,$$

where  $\zeta_n, n \geq 1$  are the mollifiers defined by (4.15). □

We next consider the integral

$$(6.6) \quad \int_0^t \int_{\mathbb{R}} f(x, s) \mathcal{L}^H(dx, ds),$$

where  $f$  is a deterministic function. For elementary function  $f_\Delta \in \mathcal{E}_*$  of the form (6.2) we define integration with respect to local time  $\mathcal{L}^H$  as follows

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} f_\Delta(x, s) \mathcal{L}^H(dx, ds) &:= \sum_{i,j} f_{ij} [\mathcal{L}^H(x_i, s_j) \\ &\quad - \mathcal{L}^H(x_i, s_{j-1}) - \mathcal{L}^H(x_{i-1}, s_j) + \mathcal{L}^H(x_{i-1}, s_{j-1})], \end{aligned}$$

for all  $t \in [0, T]$ . Notice that

$$\begin{aligned}
& \mathcal{L}^H(x_i, s_j) - \mathcal{L}^H(x_i, s_{j-1}) - \mathcal{L}^H(x_{i-1}, s_j) + \mathcal{L}^H(x_{i-1}, s_{j-1}) \\
&= [\mathcal{L}^H(x_i, s_j) - \mathcal{L}^H(x_{i-1}, s_j)] - [\mathcal{L}^H(x_i, s_{j-1}) - \mathcal{L}^H(x_{i-1}, s_{j-1})] \\
&= -[1_{(x_{i-1}, x_i]}(B^H), B^H]_{s_j}^{(W)} + [1_{(x_{i-1}, x_i]}(B^H), B^H]_{s_{j-1}}^{(W)} \\
&= -[1_{(x_{i-1}, x_i]}(B^H)1_{(s_{j-1}, s_j]}(\cdot), B^H]_t^{(W)}
\end{aligned}$$

for all  $i, j$ . We get the identity

$$(6.7) \quad \int_0^t \int_{\mathbb{R}} f_{\Delta}(x, s) \mathcal{L}^H(dx, ds) = -[f_{\Delta}(B^H, \cdot), B^H]_t^{(W)}$$

for all  $t \in [0, T]$ . Moreover, for  $f \in \mathcal{H}_*$  we can define

$$\int_0^t \int_{\mathbb{R}} f(x, s) \mathcal{L}^H(dx, ds) := \lim_{n \rightarrow \infty} \int_0^t \int_{\mathbb{R}} f_{\Delta, n}(x, s) \mathcal{L}^H(dx, ds), \quad \text{in } L^2$$

for all  $t \in [0, 1]$  if  $f_{\Delta, n} \rightarrow f$  in  $\mathcal{H}_*$ , as  $n$  tends to infinity, where  $\{f_{\Delta, n}\} \subset \mathcal{E}_*$ .

**Theorem 6.3.** *Let  $0 < H < \frac{1}{2}$  and  $f \in \mathcal{H}_*$ . Then the integral (6.6) exists in  $L^2$  and the Bouleau-Yor identity takes the form*

$$(6.8) \quad [f(B^H, \cdot), B^H]_t^{(W)} = - \int_0^t \int_{\mathbb{R}} f(x, s) \mathcal{L}^H(dx, ds)$$

for all  $t \in [0, T]$ .

**Corollary 6.1.** *Let  $0 < H < \frac{1}{2}$ ,  $F \in C^{1,1}(\mathbb{R} \times \mathbb{R}_+)$  and  $\frac{\partial}{\partial x} F = f \in \mathcal{H}_*$ . Then the Itô type formula*

$$\begin{aligned}
F(B_t^H, t) &= F(0, 0) + \int_0^t f(B_s^H, s) dB_s^H \\
&\quad + \int_0^t \frac{\partial}{\partial t} F(B_s^H, s) ds - \frac{1}{2} \int_0^t \int_{\mathbb{R}} f(x, s) \mathcal{L}^H(dx, ds)
\end{aligned}$$

holds.

Finally, let us consider the weighted local time of fBm  $B^H$  with  $0 < H < \frac{1}{2}$  on a continuous curve. Let  $a(t)$  denote a continuous function on  $[0, T]$ . Then the function

$$f_a(x, s) = 1_{(-\infty, a(s))}(x)$$

belongs to  $\mathcal{H}_*$ , and the integral

$$\int_0^t \int_{\mathbb{R}} f_a(x, s) \mathcal{L}^H(dx, ds)$$

and the generalized quadratic covariation  $[f_a(B^H, \cdot), B^H]^{(W)}$  exist in  $L^2$ . By the idea due to Eisenbaum [7] and Föllmer *et al* [10], as an example, we can show that the process

$$\int_0^t \int_{\mathbb{R}} f_a(x, s) \mathcal{L}^H(dx, ds), \quad t \geq 0$$

is increasing and continuous. Thus, we can define the weighted local time of  $B^H$  with  $0 < H < \frac{1}{2}$  at a continuous curve  $t \mapsto a(t)$  by setting

$$\mathcal{L}^H(a(\cdot), t) = \int_0^t \int_{\mathbb{R}} f_a(x, s) \mathcal{L}^H(dx, ds).$$

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